12th Benelux Mathematical Olympiad

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Problems and Solutions

Problem Selection Committee
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Problem 1

Find all positive integers d with the following property: there exists a polynomial P of degree d with integer coefficients such that $|P(m)| = 1$ for at least $d+1$ different integers m.

Solution

Note that $P(x) = c$ for a fixed constant has at most d solutions, since the polynomial $P(x) - c$ of degree d cancels at most d times. This implies that there are integers m satisfying $P(m) = 1$, as well as integers m such that $P(m) = -1$.

Next, we prove the following lemma.

Lemma. If a and b are integers such that $P(a) = -1$ and $P(b) = 1$, then $|b - a| \leq 2$.

Proof Since $b - a \mid P(b) - P(a)$ by a well-known lemma (corollary of $a - b \mid a^n - b^n$ for every integer $n \geq 0$), the conclusion follows.

Let us first consider the case that $d \geq 4$, and assume that there exists a polynomial P with at least $d + 1 \geq 5$ solutions to $|P(m)| = 1$. Let a and b be the smallest and largest solution respectively. Since $b - a \geq 4$, we need $P(a) = P(b)$ by the lemma. Without loss of generality (by switching P with $-P$ if necessary) we can assume $P(a) = P(b) = 1$. Take a value m such that $P(m) = -1$. Due to the lemma, we need $b-m$ and $m-a$ to be both at most 2. Since $b-a \geq 4$, there is only one possibility left in which case $b-a = 4$ and thus $d = 4$. By considering $P(x - m)$, we can assume $P(\pm 2) = P(\pm 1) = 1$ and $P(0) = -1$. The unique fourth degree polynomial satisfying these equalities is $P(x) = -0.5(x^2 - 4)(x^2 - 1) + 1$ which is not a polynomial with integer coefficients.

For $1 \leq d \leq 3$, there exist polynomials satisfying the conditions.

For $d = 1$ we can take $P_1(X) = X$ as $P_1(-1) = -1$ and $P_1(1) = 1$.

For $d = 2$, $P_2(X) = 2X(X - 2) + 1$ satisfies $P_2(0) = 1, P_2(1) = -1$ and $P_2(2) = 1$.

For $d = 3$, the polynomial ; $P_3(X) = (X + 1)X(X - 2) + 1$ satisfies $P_3(-1) = 1 = P_3(0) = P_3(2) = 1$ and $P_3(1) = -1$. So $|P_3(m)| = 1$ for $m \in \{-1, 0, 1, 2\}$.

Thus the integers with the required property are precisely $d = 1, 2, 3$. This completes the proof. ☐

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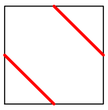
Problem 2

Let N be a positive integer. A collection of $4N^2$ unit tiles with two segments drawn on them as shown is assembled into a $2N \times 2N$ board. Tiles can be rotated.



The segments on the tiles define paths on the board. Determine the least possible number and the largest possible number of such paths.

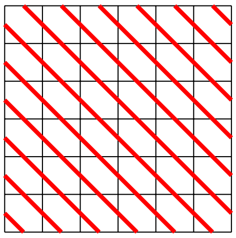
Solution

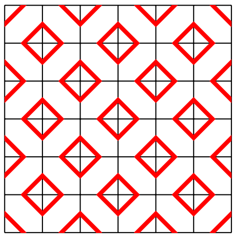
Let p denote the number of paths. Notice that there are two types of paths: (1) those that start and end at a point on the boundary of the board and (2) closed paths in the interior of the board. Let p_1, p_2 denote the respective numbers of paths of either type. There are $8N$ points on the boundary of the board, and each of these is the starting point or endpoint of exactly one path, so $p_1 = 4N$. Trivially, $p_2 \geqslant 0$, so $p = p_1 + p_2 \geqslant 4N$.

The paths on the board are made up of $8N^2$ segments in total. There are only 4 possible paths of one segment, in the corners of the board. All other paths on the boundary of the board therefore consist of at least 2 segments. Moreover, all closed paths in the interior of the board consist of at least 4 segments. Hence

$$8N^2 \geqslant 4 \cdot 1 + (p_1 - 4) \cdot 2 + p_2 \cdot 4 \iff p_2 \leqslant N^2 + (N-1)^2, \text{ so } p = p_1 + p_2 \leqslant N^2 + (N+1)^2.$$

We have thus shown that $4N \leqslant p \leqslant N^2 + (N+1)^2$. These minimum and maximum values can indeed be attained, as shown below for $N = 3$.

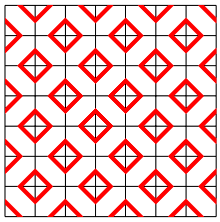




The constructions generalise easily.The constructions clearly attain the required bounds because they satisfy the equalities in our arguments showing the two bounds. Indeed the board on the left has $p_2 = 0$ so give the lower bound. The one on the right has exactly 4 boundary paths with one segment, all other $4N - 4$ boundary paths with 2 segments, and all remaining paths with 4 segments. □

Remark. The same ideas solve the analogous problem for a $(2N+1) \times (2N+1)$ assembled from $(2N+1)^2$ such tiles. For this board, $p \geqslant p_1 = 2(2N+1)$. Next, there are $2(2N+1)^2$ segments, so, again,

$$2(2N+1)^2 \geqslant 4 \cdot 1 + (p_1 - 4) \cdot 2 + p_2 \cdot 4 \iff p_2 \leqslant 2N^2 + \frac{1}{2}.$$



But p_2 is an integer, so $p_2 \leqslant 2N^2$, and hence $p \leqslant 2(N+1)^2$. We have thus shown that $2(2N+1) \leqslant p \leqslant 2(N+1)^2$. The construction for the lower bound is the same as for the $2N \times 2N$ board; the construction for the upper bound, shown for $N = 3$, is a small modification of that for the $2N \times 2N$ board and again generalises easily. □

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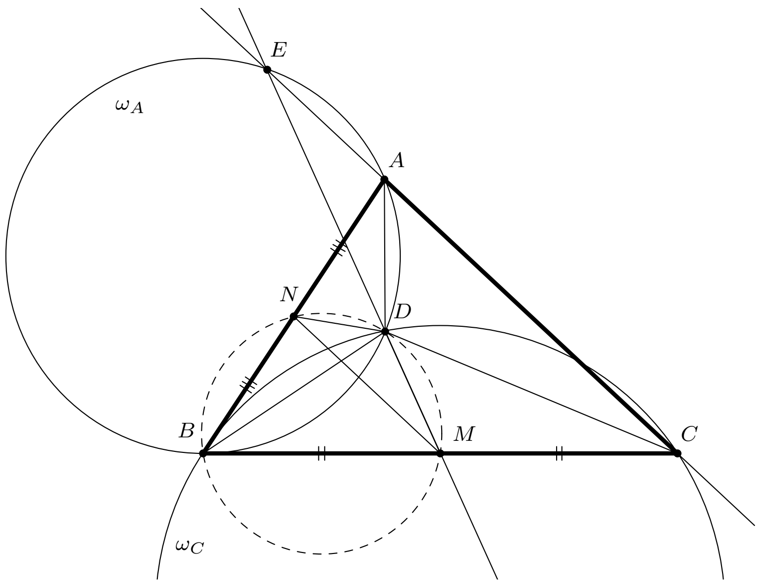
Problem 3

Let ABC be a triangle. The circle ω_A through A is tangent to line BC at B. The circle ω_C through C is tangent to line AB at B. Let ω_A and ω_C meet again at D. Let M be the midpoint of line segment $[BC]$, and let E be the intersection of lines MD and AC. Show that E lies on ω_A.

Solution 1

Let N be the midpoint of $[AB]$. By tangential angles, $\angle CBD = \angle BAD$ and $\angle DBA = \angle DCB$, so triangles DAB and DBC are similar. By definition of the midpoints M, N, so are triangles AND and BMD. In particular, $\angle BND = 180° - \angle DNA = 180° - \angle DMB$, so $BNDM$ is cyclic. But $MN \parallel AC$ by construction, so $\angle DBA = \angle DBN = \angle DMN = \angle EMN = \angle MEA = \angle DEA$, hence $EADB$ is cyclic, completing the proof. □



Solution 2

Let S be the point on ω_C such that BS is parallel to AC, and let E' be the reflection of S in M (such that $BSCE'$ is a parallellogram and E' lies on AC). Now we do some (directed) angle chasing:

$$\begin{aligned}\angle AE'B &= \angle CE'B \text{ (since } A, E' \text{ and } C \text{ are collineair)}\\ &= \angle BSC \text{ (} BSCE' \text{ is a parallellogram)}\\ &= \angle ABC \text{ (inscribed angles on } \omega_C \text{)}\\ &= \angle ADB \text{ (inscribed angles on } \omega_A\text{).}\end{aligned}$$

Hence E' lies on ω_A. Further $\angle AE'D = \angle ABD = \angle BSD$, and because AE' is parallel to BS we find that $E'D$ is parallel to SD. This means that E' lies on MD, so $E' = E$ and E lies on ω_A. □

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Problem 4

A divisor d of a positive integer n is said to be a *close* divisor of n if $\sqrt{n} < d < 2\sqrt{n}$. Does there exist a positive integer with exactly 2020 close divisors?

Solution

Let m be an odd integer with exactly 2020 positive divisors and which is (automatically) not a square. For example, $m = 3^{2019}$ suffices, but of course there are many alternatives. Now consider $n = 2^k m$, for some integer k such that $2^k > m$. Any divisor of n is then of the form $2^\ell d$ where d is a divisor of m. We will now show that for every such divisor d, there exists a unique ℓ such that $2^\ell d$ is a close divisor. Because $\sqrt{n}$ is not an integer, there certainly is a unique integer a such that $\sqrt{n} < 2^a d < 2\sqrt{n}$. Because $2^k > m$, we have $m < \sqrt{n} < 2^k$. Combining with $1 \leq d \leq m$, we find $1 < \frac{\sqrt{n}}{d} < 2^a < \frac{2\sqrt{n}}{d} < 2 \cdot 2^k$ and so we see that $0 < a \leqslant k$. So $2^a d$ is indeed a close divisor. Because m has exactly 2020 divisors, we find that n has exactly 2020 close divisors. □