8th Benelux Mathematical Olympiad

Soest, 29 April – 1 May 2016

Solutions

Problem 1. Find the greatest positive integer N with the following property: there exist integers $x_1, \ldots, x_N$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$.

Solution. We prove that the greatest N with the required property is $N = 1000$. First note that $x_i^2 - x_i x_j = x_i(x_i - x_j)$, and that the prime factorisation of 1111 is $11 \cdot 101$.

We first show that we can find 1000 integers $x_1, x_2, \ldots, x_{1000}$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$. Consider the set $\{1, 2, \ldots, 1110\}$. This set contains 10 integers divisible by 101, and it contains 100 integers divisible by 11. None of the integers in the set are divisible by both 11 and 101. If we delete all of these $10 + 100$ integers from the set, we are left with 1000 integers. Call these $x_1, x_2, \ldots, x_{1000}$. Now we have $11 \nmid x_i$ and $101 \nmid x_i$ for all i. Suppose there are $i \neq j$ with $1111 \mid x_i(x_i - x_j)$, then we must have $1111 \mid x_i - x_j$, which is a contradiction, since $x_i, x_j \in \{1, 2, \ldots, 1110\}$. So this set satisfies the requirement.

We now prove that given 1001 (or more) integers $x_1, x_2, \ldots, x_{1001}$ there are $i \neq j$ with $1111 \mid x_i(x_i - x_j)$. Suppose for a contradiction that for all indices $i \neq j$, we have that $x_i(x_i - x_j)$ is not divisible by 1111, and write $X = \{x_1, \ldots, x_{1001}\}$. We may (after reducing modulo 1111 if necessary) assume that $x_i \in \{0, 1, \ldots, 1110\}$ for all i. Then we know that $x_i \neq 0$ for all i, and $x_i \neq x_j$ for all $i \neq j$. Suppose for some i we have $11 \mid x_i$. (Since $x_i \neq 0$, we know that $101 \nmid x_i$.) Then any integer $a \neq x_i$ with $a \equiv x_i \bmod 101$ cannot be an element of X, since $1111 \mid x_i(x_i - a)$. In $\{1, 2, \ldots, 1110\}$ there are 10 such integers, all of them coprime with $11 \cdot 101$. If there are exactly k different values of i such that $11 \mid x_i$, there are $10k$ different integers from $\{1, 2, \ldots, 1110\}$ that cannot be elements of X, all of them coprime with $11 \cdot 101$. Similarly, if there are exactly m different values of i such that $101 \mid x_i$, then there are $100m$ different integers from $\{1, 2, \ldots, 1110\}$ that cannot be elements of X, all of them coprime with $11 \cdot 101$. (Note that those $10k$ and $100m$ integers can overlap.)

In $\{1, 2, \ldots, 1110\}$ there are 100 multiples of 11, there are 10 multiples of 101 and there is no multiple of $11 \cdot 101$, so there are 1000 integers that are coprime with $11 \cdot 101$. In X we have $1001 - k - m$ integers that are coprime with $11 \cdot 101$, so exactly $k + m - 1$ of the coprime integers in $\{1, 2, \ldots, 1110\}$ are not in X. This implies that $10k \leqslant k + m - 1$ and $100m \leqslant k + m - 1$. Adding these two inequalities we find $8k + 98m \leqslant -2$, a clear contradiction. So $N < 1001$. □

Problem 2. Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n.

Solution. Let d_1 and d_2 be positive divisors of n that form a pair as given in the problem. If d_1 and d_2 have a non-trivial prime divisor p in common, then $p \mid d_1+d_2$ and $p \leqslant d_1 < d_1+d_2$, so $d_1 + d_2$ cannot be prime. Hence $\gcd(d_1, d_2) = 1$, which implies that $d_1 d_2 \mid n$. Suppose the number of positive divisors of n is $2t$ (it is even since the divisors can be split into pairs). If we now multiply all divisors, then on one hand we have the product of all $d_1 d_2$ where $\{d_1, d_2\}$ is a pair, so that product is at most n^t. On the other hand for every divisor d there is another divisor $\frac{n}{d}$ (since the number of divisors is even, the case $d = \frac{n}{d}$ does not occur), and the product of all those is equal to n^t. Hence there must be equality in every inequality $d_1 d_2 \leqslant n$. So the pairs of divisors given in the problem are all of the form $\{d, \frac{n}{d}\}$.

Now we prove the two statements in the problem. Suppose that d, d' are positive divisors of n such that $d + \frac{n}{d} = d' + \frac{n}{d'}$. Then $d^2 d' + nd' = d(d')^2 + nd$, so $dd'(d - d') = n(d - d')$ and hence $(dd' - n)(d - d') = 0$. Therefore either $d = d'$ or $dd' = n$, which implies that $\{d, \frac{n}{d}\} = \{d', \frac{n}{d'}\}$, as required.

Now let d be a positive divisor of n. Every prime divisor p of n divides precisely one of d and $\frac{n}{d}$, since $d\frac{n}{d} = n$ and $\gcd(d, \frac{n}{d}) = 1$; so $p \nmid d + \frac{n}{d}$. Therefore $\gcd(n, d + \frac{n}{d}) = 1$. Since $d + \frac{n}{d} > 1$ we conclude that $d + \frac{n}{d}$ cannot be a divisor of n. $\square$

Problem 3. Find all functions $f\colon \mathbb{R} \to \mathbb{Z}$ such that

$$\Big(f\big(f(y)-x\big)\Big)^2 + f(x)^2 + f(y)^2 \;=\; f(y)\cdot\Big(1+2f\big(f(y)\big)\Big)$$

for all $x, y \in \mathbb{R}$.

Solution I. Take $x = y = 0$ and write $c = f(0)$, then we find $f(c)^2 + c^2 + c^2 = c + 2cf(c)$, so $(f(c)-c)^2 = c - c^2$. The left-hand side is non-negative, so the right-hand side must be non-negative as well, hence $c - c^2 \geqslant 0$, so $c(1-c) \geqslant 0$. This implies $0 \leqslant c \leqslant 1$, and since $c \in \mathbb{Z}$, we get $c = 0$ or $c = 1$. In both cases we have $c - c^2 = 0$, so $f(c) - c = 0$, hence $f(c) = c$. Now taking $y = 0$ we find for all $x \in \mathbb{R}$ that

$$\big(f(c-x)\big)^2 + f(x)^2 + c^2 = c + 2c^2. \tag{1}$$

If $c = 0$, then this equation reduces to $f(-x)^2 + f(x)^2 = 0$, and since the left-hand side consists of the sum of two squares, both squares must be zero. Therefore $f(x) = 0$ for all x. This function is indeed a solution of the given functional equation.

Now we consider the other case: $c = 1$. Then (1) reduces to $f(1-x)^2 + f(x)^2 = 2$. The left-hand side consists of two squares of integers, so they must both be 1. Therefore for all x we have $f(x) = 1$ or $f(x) = -1$.

Suppose there is an a with $f(a) = -1$. We take $y = a$ in the functional equation; using that $f(w)^2 = 1$ for any w, we find $1+1+1 = -1\cdot(1+2f(-1))$. Both with $f(-1) = 1$ and with $f(-1) = -1$ this gives a contradiction. So there exists no such a, and we conclude that $f(x) = 1$ for all $x \in \mathbb{R}$. This is also a solution of the given functional equation.

We conclude that there are two solutions: $f(x) = 0$ for all x, and $f(x) = 1$ for all x. $\square$

Solution II. We proceed as in the first solution up to the point of deriving $f(x) = \pm 1$ for all $x \in \mathbb{R}$. Now we take $x = 0$ in the original functional equation, and we find

$$\Big(f\big(f(y)\big)\Big)^2 + f(0)^2 + f(y)^2 \;=\; f(y)\cdot\Big(1+2f\big(f(y)\big)\Big).$$

We can rewrite this as

$$\Big(f(y) - f\big(f(y)\big)\Big)^2 + f(0)^2 = f(y).$$

The left-hand side is non-negative, so $f(y) \geqslant 0$ for all y. If we combine this with $f(x) = \pm 1$ for all x, we can conclude that $f(x) = 1$ for all $x \in \mathbb{R}$. And this is a solution. $\square$

Problem 4. A circle ω passes through the two vertices B and C of a triangle ABC. Furthermore, ω intersects segment AC in $D \neq C$ and segment AB in $E \neq B$. On the ray from B through D lies a point K such that $|BK| = |AC|$, and on the ray from C through E lies a point L such that $|CL| = |AB|$. Show that the circumcentre O of triangle AKL lies on ω.

Solution I. Let M be the midpoint of the arc BC of ω that is on the same side of BC as A. Then $|BM| = |CM|$. We also have $|BA| = |CL|$ and $\angle ABM = \angle EBM = \angle ECM = \angle LCM$. Hence $\triangle ABM \cong \triangle LCM$. So $|AM| = |LM|$. (In case $M = E$ the triangles ABM and LCM are degenerate, but then the proof still works, since $|AM| = |AB| - |BM| = |LC| - |CM| = |LM|$.) Similarly, we prove that $|AM| = |KM|$, so M is the circumcentre of $\triangle AKL$. This means that $M = O$, and hence we are done as M was defined to be on ω. □

Solution II. We consider the configuration where O is in the interior of triangle ABC. The proof for other configurations is similar. Furthermore, we exclude the case that $O = D$ or $O = E$; in those cases it is immediate that O is on ω.

We have $\angle ABK = \angle EBD = \angle ECD = \angle LCA$. Together with $|AB| = |CL|$ and $|AC| = |BK|$ this implies $\triangle ABK \cong \triangle LCA$. Hence $|AK| = |AL|$, $\angle AKB = \angle LAC$ (denote this angle by α) and $\angle BAK = \angle CLA$ (denote this angle by β). Furthermore, let $\gamma = \angle BEC = \angle BDC$. Then $\angle BAC = 180° - \angle BDA - \angle ABD = \angle BDC - \angle ABD = \gamma - \angle ABD$. Therefore $\angle KAL = \angle BAK + \angle LAC - \angle BAC = \alpha + \beta - (\gamma - \angle ABD) = \alpha + \beta + \angle ABD - \gamma$. Note that in triangle ABK we have $\alpha + \beta + \angle ABD = 180°$, hence $\angle KAL = 180° - \gamma$.

We have $\angle KOL = 2(180° - \angle KAL) = 2\gamma$. Since $|AK| = |AL|$, this implies $\angle AOL = \gamma$. As we also have $\angle AEL = \angle BEC = \gamma$, the quadrilateral $OELA$ is cyclic. Analogously, $ODKA$ is cyclic. Now we have

$$\angle DOE = 360° - \angle DOA - \angle AOE = 180° - \angle DOA + 180° - \angle AOE$$

$$= \angle DKA + \angle ALE = \angle BKA + \angle ALC = \alpha + \beta.$$

On the other hand

$$\angle DBE = \angle KBA = 180° - \angle BAK - \angle AKB = 180° - \alpha - \beta.$$

Hence $\angle DOE + \angle DBE = 180°$, so O is on the circle containing D, B and E, which is ω. □