# BxMO 2015

## *Problems with Solutions*

## *Problem 1.*

**Determine the smallest positive integer $q$ with the following property:**

> **for every integer $m$ with $1 \leqslant m \leqslant 1006$, there exists an integer $n$ such that**
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> $$\frac{m}{1007} q < n < \frac{m+1}{1008} q.$$

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**Solution 1.** For $m = 1006$, we have

$$q - q/1007 < n < q - q/1008,$$

for some integer $n$. If $q \leqslant 1007$, then $q - q/1007$ and $q - q/1008$ are both numbers that are at least $q - 1$ and smaller than $q$, so there can be no integer $n$ in between. Hence $q > 1007$ and $q - q/1008 \leqslant q - 1$, implying that $n < q - 1$, and that $q - q/1007 < q - 2$. By rearranging terms, we find $q > 2014$, and hence $q \geqslant 2015$.

Let us prove that $q = 2015$ works. Indeed, $mq/1007 = 2m + m/1007 < 2m + 1$ and $(m+1)q/1008 = 2(m+1) - (m+1)/1008 > 2m+1$ for $1 \leqslant m \leqslant 1006$. Hence each pair of inequalities can be satisfied by taking $n = 2m + 1$. This completes the proof, and shows that the smallest possible of $q$ is indeed 2015. □

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**Solution 2.** For $m = 1, \ldots, 1006$, there must exist an integer $N$ divisible by $1007 \cdot 1008$ satisfying the double inequality

$$1008mq < N < 1007(m + 1)q.$$

Since $N$ is divisible by 1007 and 1008, we may write

$$N = 1008mq + 1008k = 1007(m + 1)q - 1007\ell, \qquad (*)$$

where $k, \ell > 0$ are integers. Hence

$$(1007 - m)q = 1008k + 1007\ell \geqslant 2015,$$

since $k, \ell > 0$. Choosing $m = 1006$ in this last inequality, it follows that $q \geqslant 2015$. Conversely, for $q = 2015 = 1008 + 1007$, $(*)$ can be satisfied by taking $k = \ell = 1007 - m$. (Indeed, 1007 and 1008 are coprime, and so integer divisible by 1007 and 1008 is also divisible by their product.) Thus $q = 2015$ has the desired property, and we are done. □

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## *Problem 2.*

**Let $ABC$ be an acute triangle with circumcentre $O$. Let $\Gamma_B$ be the circle through $A$ and $B$ that is tangent to $AC$, and let $\Gamma_C$ be the circle through $A$ and $C$ that is tangent to $AB$. An arbitrary line through $A$ intersects $\Gamma_B$ again in $X$ and $\Gamma_C$ again in $Y$. Prove that $|OX| = |OY|$.**

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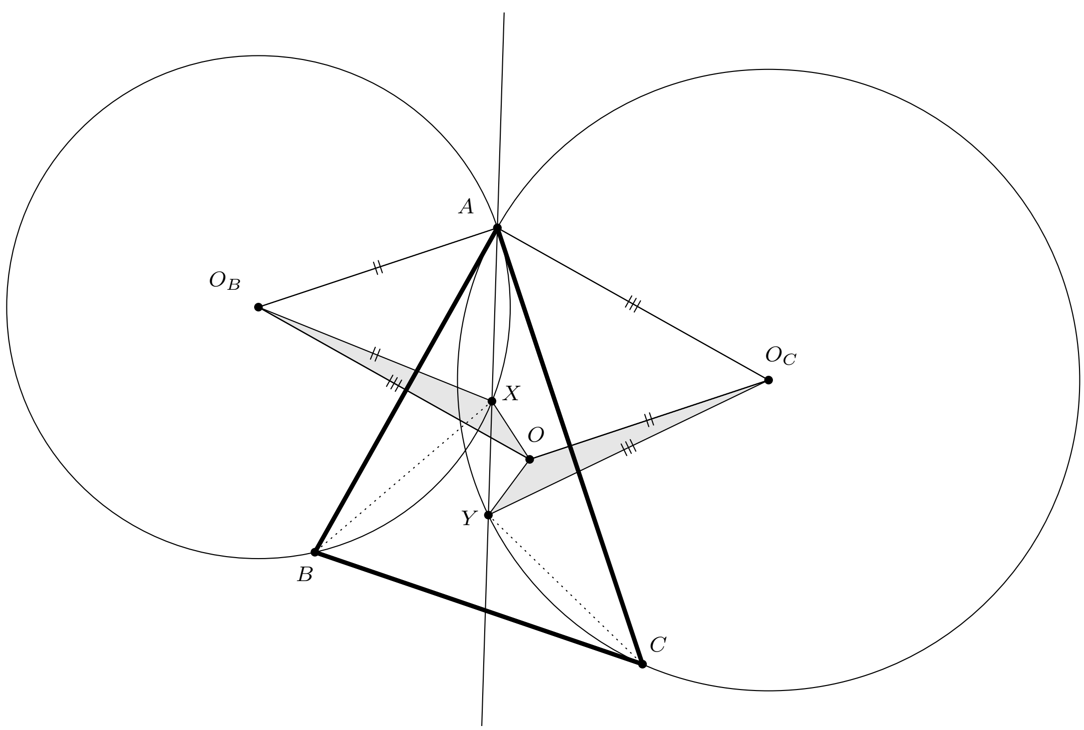
*The following solutions are valid for the configurations appearing in the diagrams.*

**Solution 1.** Let $O_B$ and $O_C$ denote the respective centres of $\Gamma_B$ and $\Gamma_C$. We shall show that $XO_BO$ and $OO_CY$ are congruent. Now $AO_B \perp AC$ since $\Gamma_B$ is tangent to $AC$ and $OO_C \perp AC$ since $OO_C$ is the perpendicular bisector of $[AC]$. Hence $AO_B \parallel OO_C$, and similarly, $AO_C \parallel OO_B$. It follows that $AO_BOO_C$ is a parallelogram. In particular, $|O_BX| = |AO_B| = |OO_C|$ and $|O_CY| = |O_CA| = |OO_B|$.

It will therefore suffice to show that the angles $\angle OO_BX$ and $\angle YO_CO$ are equal. Noting that $\angle AO_BO = \angle AO_CO$ since $AO_BOO_C$ is a parallelogram, this follows by angle chasing:

$$\begin{aligned}\angle XO_BO &= \angle AO_BO - \angle AO_BX = \angle AO_CO - \Big(180^\circ - 2\angle O_BAX\Big)\\ &= \angle AO_CO - 180^\circ + 2\Big(\angle O_BAO_C - \angle YAO_C\Big)\\ &= \angle AO_CO - 180^\circ + 2\Big(180^\circ - \angle AO_CO\Big) - \Big(180^\circ - \angle AO_CY\Big)\\ &= \angle AO_CY - \angle AO_CO = \angle OO_CY.\end{aligned}$$

□



**A Variant.** As in the main solution, we note that $AO_BOO_C$ is a parallelogram. Notice that $O_B$ and $O_C$ lie on the respective perpendicular bisectors of $[AX]$ and $[AY]$. The following lemma then implies that $O$ lies on the perpendicular bisector of $[XY]$, completing the proof: □

> Let $P_1, P_2, P_3$ be three points on a line. Let $O_1$ be a point on the perpendicular bisector of $[P_2P_3]$, let $O_2$ be a point on the perpendicular bisector of $[P_3P_1]$ and let $O_3$ be the point in the plane such that $O_1P_3O_2O_3$ is a parallelogram. Then $O_3$ lies on the perpendicular bisector of $[P_1P_2]$.
>
> *Proof.* One can choose coordinates such that $P_3 = (0, 0)$ and $P_1 = (2, 0)$. Then $P_2 = (2a, 0)$, $O_1 = (a, b)$ and $O_2 = (1, c)$ for some $a, b, c \in \mathbb{R}$. Hence $O_3 = (a + 1, b + c)$ lies on the perpendicular bisector of $[P_1P_2]$. □

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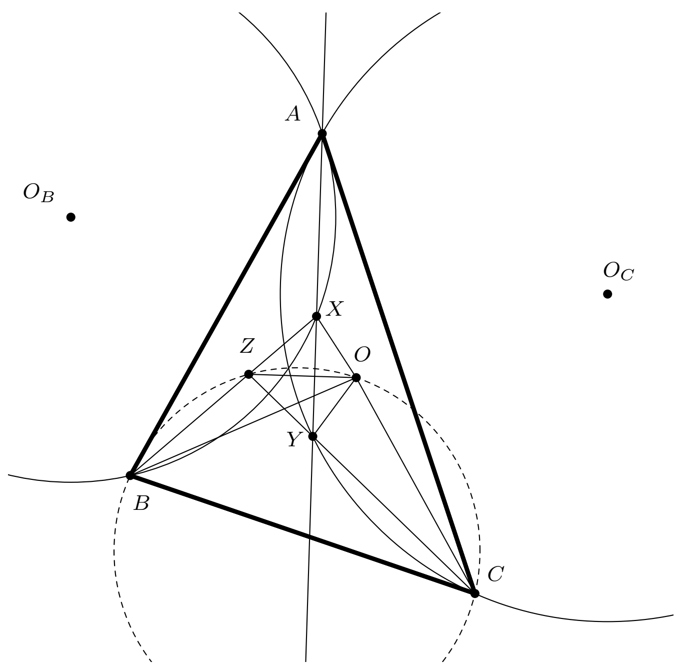
**Solution 2.** Let $\alpha = \angle BAC$. Observe that $\angle AXB = 180° - \alpha = \angle CYA$ by tangential angles. Let $Z$ be the intersection of $BX$ and $CY$. Thus $ZXY$ is isosceles with, in particular, $\angle ZXY = \angle ZYX = \alpha$. It will thus suffice to show that $OZ$ bisects $\angle XZY$.

Now $\angle BZC = 2\alpha = \angle BOC$ since $O$ is the circumcentre of $ABC$, and hence $BZOC$ is cyclic. In particular, since $|OB| = |OC|$, it follows that $\angle OZY = \angle OBC = 90° - \alpha$. But $\angle XZY = 180° - 2\alpha$, and so $\angle XZY = 2\angle OZY$, which shows that $OZ$ bisects $\angle XZY$. □

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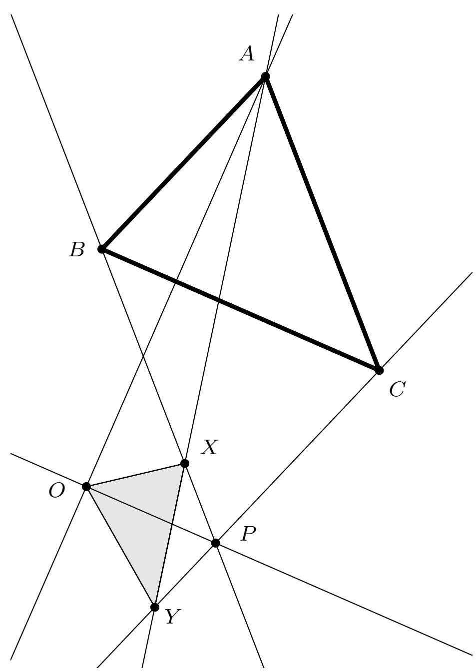
**Solution 3.** Consider inversion $\mathscr{I}$ in a circle centred at $A$. Under $\mathscr{I}$,

$$B \mapsto B', \qquad C \mapsto C', \qquad O \mapsto O', \qquad X \mapsto X', \qquad Y \mapsto Y',$$
$$\Gamma_B \mapsto \gamma_B, \text{ a line through } B' \text{ parallel to } AC',$$
$$\Gamma_C \mapsto \gamma_C, \text{ a line through } C' \text{ parallel to } AB',$$

Notice that $\mathscr{I}$ sends the circumcircle of $ABC$ to the line $B'C'$, and hence maps $AO$ to a line perpendicular to $BC$. Further, if $[AD]$ is a diameter of this circumcircle and $D \mapsto D'$ under $\mathscr{I}$, then $D'$ lies on $BC$. Also, $|AD| = 2|AO|$ implies $|AO'| = 2|AD'|$, and hence $O'$ is the image of $A$ under reflection in $BC$. Finally, $\angle OXA = \angle X'O'A$ and $\angle OYA = \angle Y'O'A$, and hence $|OX| = |OY|$ if and only if $O'A$ bisects $\angle X'O'Y'$ externally. We have thus reduced the problem to the following statement:

> In triangle $ABC$, let $O$ be the reflection of $A$ over $BC$, $\gamma_B$ be a line parallel to $AC$ through $B$, and $\gamma_C$ be a line parallel to $AB$ through $C$. For an arbitrary line $\ell$ passing through $A$, let $X$ and $Y$ be the intersections of $\ell$ with $\gamma_B$ and $\gamma_C$, respectively. Prove that $OA$ bisects $\angle XOY$ externally.




Let $P = \gamma_A \cap \gamma_B$. By construction, $POBC$ is an isosceles trapezoid (and therefore cyclic), $\angle OBX = \angle OBP = \angle OCP = \angle OCY$. Further, since $XB \parallel AC$ and $YC \parallel AB$, we have $\triangle ABX \sim \triangle YCA$. Therefore, as $|OB| = |AB|$ and $|AC| = |OC|$ by construction,

$$\frac{|OB|}{|BX|} = \frac{|AB|}{|BX|} = \frac{|YC|}{|CA|} = \frac{|YC|}{|CO|}.$$

It follows that $\triangle OBX \sim \triangle YCO$. Hence

$$\frac{|OX|}{|OY|} = \frac{|XB|}{|OC|} = \frac{|XB|}{|AC|} = \frac{|XA|}{|AY|},$$

and so the result follows from the angle bisector theorem. $\square$

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## *Problem 3.*

**Does there exist a prime number whose decimal representation is of the form $3811\cdots11$ (that is, consisting of the digits 3 and 8 in that order followed by one or more digits 1)?**

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**Solution.** Write

$$a(n) = 38\underbrace{11\cdots11}_{n \text{ digits } 1}.$$

There are three cases to consider, depending on the remainder of $n$ upon division by three.

- If $n = 3k+1 \equiv 1 \pmod 3$, then the sum of the digits of $a(n)$ is equal to $3(k+4)$, i.e. divisible by 3, and hence so is $a(n)$.
- If $n = 3k+2 \equiv 2 \pmod 3$, then note that $a(2) = 3811 = 3700+111$ is divisible by 37. By induction, as $a(3k+2) = 1000a(3k-1)+111$, it follows that $a(3k+2)$ is divisible by 37 for each $k \geqslant 0$.
- If $n = 3k \equiv 0 \pmod 3$, observe that

  $$9a(3k) = 342\underbrace{99\cdots99}_{n \text{ digits } 9} = \left(7\cdot 10^k\right)^3 - 1$$

  which is properly divisible by $7\cdot 10^k - 1$, a number that is larger than 9. Hence $a(3k)$ admits a non-trivial factor and so is not prime. □

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## *Problem 4.*

An *arithmetic progression* is a set of the form $\{a, a+d, \ldots, a+kd\}$, where $a, d, k$ are positive integers and $k \geqslant 2$. Thus an arithmetic progression has at least three elements and the successive elements have difference $d$, called the *common difference* of the arithmetic progression.

**Let $n$ be a positive integer. For each partition of the set $\{1, 2, \ldots, 3n\}$ into arithmetic progressions, we consider the sum $S$ of the respective common differences of these arithmetic progressions. What is the maximal value $S$ that can attain?**

(A *partition* of a set $A$ is a collection of disjoint subsets of $A$ whose union is $A$.)

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**Solution.** The maximum value is $n^2$, which is attained for the partition into $n$ arithmetic progressions $\{1, n+1, 2n+1\}, \ldots, \{n, 2n, 3n\}$, each of difference $n$.

Suppose indeed that the set has been partioned into $N$ progressions, of respective lengths $\ell_i$, and differences $d_i$, for $1 \leqslant i \leqslant N$. Since $\ell_i \geqslant 3$,

$$2\sum_{i=1}^{N} d_i \leqslant \sum_{i=1}^{N} (\ell_i - 1)d_i = \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} b_i,$$

where $a_i$ and $b_i$ denote, respectively, the largest and smallest elements of progression $i$. Now

$$\sum_{i=1}^{N} b_i \geqslant 1 + 2 + \cdots + N = N(N+1)/2,$$
$$\sum_{i=1}^{N} a_i \leqslant (3n - N + 1) + \cdots + 3n = N(6n - N + 1)/2,$$

and thus

$$2\sum_{i=1}^{N} d_i \leqslant N(3n - N) \leqslant 2n^2$$

as $N(3n - N)$ is an increasing of $N$ on the interval $[0, 3n/2]$ and since $N \leqslant n$. This completes the proof. □

**Remark.** The maximising partition is in fact unique: from the above, it is immediate that all maximal partitions must have $n_i = 3$ and hence $n = N$. It follows that the minimal and maximal elements of the arithmetic progressions of a maximal partition are precisely $1, 2, \ldots, n$ and $2n+1, 2n+2, \ldots, 3n$, respectively. Consider the progression $\{n+1-d, n+1, n+1+d\}$ of difference $d$. Then $n+1-d \geqslant 1$ and $n+1+d \geqslant 2n+1$, which implies $d = n$, and hence $\{1, n+1, 2n+1\}$ is an element of any maximal partition. By induction, it follows similarly that $\{k, n+k, 2n+k\}$ is an element of any maximal partition for $1 \leqslant k \leqslant n$, since integers allowing for smaller or larger differences have already been used up. This proves the partition $\{1, n+1, 2n+1\}, \ldots, \{n, 2n, 3n\}$ is the unique maximising partition, as claimed.