6th Benelux Mathematical Olympiad
Solutions

Brugge, May 2–4 2014

Problem 1

Find the smallest possible value of the expression

$$\left\lfloor \frac{a+b+c}{d} \right\rfloor + \left\lfloor \frac{b+c+d}{a} \right\rfloor + \left\lfloor \frac{c+d+a}{b} \right\rfloor + \left\lfloor \frac{d+a+b}{c} \right\rfloor,$$

in which a, b, c and d vary over the set of positive integers.

(Here $\lfloor x \rfloor$ denotes the biggest integer which is smaller than or equal to x.)

Solution

The answer is 9.

Notice that $\lfloor x \rfloor > x - 1$ for all $x \in \mathbb{R}$. Therefore the given expression is strictly greater than

$$\frac{a+b+c}{d} + \frac{b+c+d}{a} + \frac{c+d+a}{b} + \frac{d+a+b}{c} - 4,$$

which can be rewritten as

$$\left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{a}{c} + \frac{c}{a}\right) + \left(\frac{a}{d} + \frac{d}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{b}{d} + \frac{d}{b}\right) + \left(\frac{c}{d} + \frac{d}{c}\right) - 4.$$

Since $t + \frac{1}{t} \geq 2$ for $t > 0$, we get that $6 \cdot 2 - 4 = 8$ is a *strict* lower bound for the given expression; since it takes integral values only, we actually get that 9 is a lower bound.

It remains to check that 9 can be attained; this happens for $a = b = c = 5$ and $d = 4$.

Problem 2

Let $k \geq 1$ be an integer.

We consider $4k$ chips, $2k$ of which are red and $2k$ of which are blue. A sequence of those $4k$ chips can be transformed into another sequence by a so-called *move*, consisting of interchanging a number (possibly one) of consecutive red chips with an equal number of consecutive blue chips. For example, we can move from $r\underline{bb}br\underline{rr}b$ to $r\underline{rr}br\underline{bb}b$ where r denotes a red chip and b denotes a blue chip.

Determine the smallest number n (as a function of k) such that starting from any initial sequence of the $4k$ chips, we need at most n moves to reach the state in which the first $2k$ chips are red.

Solution

The answer is $n = k$.

We will first show that $n \geq k$. Let us count the number C of times a red chip is directly to the right of a blue chip. In the final position this number equals 0. In the position $brbrbr \cdots br$ this number equals $2k$. We claim that any move reduces this number by at most 2. Denote by R the group of red chips and by B the group of blue chips that are interchanged. Any reduction in C must involve a red chip getting rid of its blue left neighbour or a blue chip getting rid of its red right neighbour. This can only happen with the leftmost chip of R (if its left neighbour is blue) and the rightmost chip of B (if its right neighbour is red), but not with the rightmost chip of R (and its right neighbour) or the leftmost chip of B (and its left neighbour). Hence C is reduced by at most 2 in any move. Therefore the number of moves necessary to change $brbrbr \cdots br$ into the final position is at least $\frac{2k}{2} = k$.

We will now show that $n \leq k$, i.e. that it is always possible to perform at most k moves in order to reach the state in which the first $2k$ chips are red. Consider the first $2k$ chips. If at most k of these chips are blue, we can perform one move for each chip, switching it with one of the red chips of the last $2k$ chips. So then we are done in at most k moves. Now suppose that of the first $2k$ chips, at least $k+1$ are blue. Then at most $k-1$ chips are red. Hence it is possible to perform at most $k-1$ moves to reach the situation in which the last $2k$ chips are red. We then perform one final move, switching the first $2k$ chips and the last $2k$ chips, ending in the situation in which the first $2k$ chips are red. Thus, it is always possible to reach the situation in which the first $2k$ chips are red in at most k steps, hence $n \leq k$.

We have now shown that $n \geq k$ and $n \leq k$, hence $n = k$ as claimed.

Problem 3

Find all positive integers $n > 1$ with the following property:

> for each two positive divisors $k, \ell < n$ of n, at least one of the numbers $2k - \ell$ and $2\ell - k$ is a (not necessarily positive) divisor of n as well.

Solution

If n is prime, then n has the desired property: if $k, \ell < n$ are positive divisors of a prime n, we have $k = \ell = 1$, in which case $2k - \ell = 1$ is a divisor of n as well.

Assume now that a composite number n has the desired property. Let p be its smallest prime divisor and let $m = n/p$; then $m \geq p \geq 2$. Choosing $(k, \ell) = (1, m)$, we see that at least one of $2m - 1$ and $m - 2$ must divide n. However, $m < 2m - 1 < 2m \leq n$; since m and n are the two biggest positive divisors of n, we conclude that $2m-1$ cannot divide n. Therefore n must be divisble by $m - 2$. It follows that $m - 2 \mid mp - p(m - 2) = 2p$, hence $m - 2 \in \{1, 2, p, 2p\}$.

We deal with each case separately.

- If $m - 2 = 1$ we have $m = 3$. As $p \leq m$, we have $p \in \{2, 3\}$, hence $n = 6$ or $n = 9$.
- If $m - 2 = 2$ we have $m = 4$, hence n is even and $p = 2$, so $n = 8$.
- If $m-2 = p$, we may assume that $p > 2$ (since we already discussed the case $m-2 = 2$). We have $m = p + 2$, hence $n = p(p + 2)$. Applying the condition in the problem to the pair $(k, \ell) = (1, p)$, we get that $p - 2$ or $2p - 1$ divides n. If $p - 2$ divides n, we must have $p = 3$ since p is the smallest prime divisor of n. If $2p - 1$ divides n, we must have $p + 2 = 2p - 1$, since $p + 2 \leq 2p - 1 < p(p + 2)$ and $p + 2$ and $p(p + 2)$ are the biggest positive divisors of n. Therefore $p = 3$ in both cases, and we get $n = 15$ as a candidate.
- If finally $m - 2 = 2p$ we have $m = 2p + 2$, so n is even. Hence $p = 2$ and $n = 12$.

We conclude that if a composite number n satisfies the condition in the problem statement, then $n \in \{6, 8, 9, 12, 15\}$. Choosing $(k, \ell) = (1, 2)$ shows that $n = 8$ does not work; choosing $(k, \ell) = (3, 6)$ shows that 12 is not a solution. It is easy to check that $n = 6$, $n = 9$ and $n = 15$ have the desired property by checking the condition for all possible pairs (k, ℓ).

We conclude that the solutions are given by the prime numbers, $n = 6$, $n = 9$ and $n = 15$.

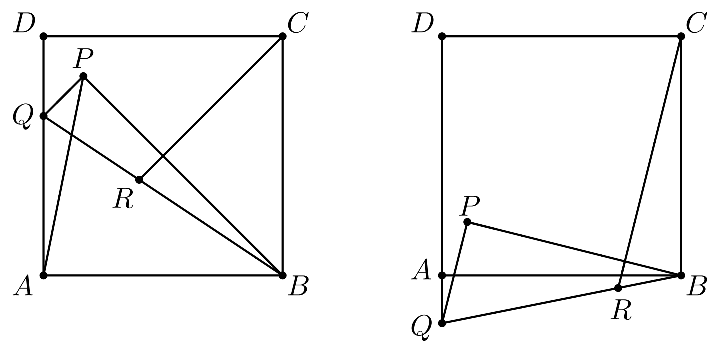
Problem 4

Let $ABCD$ be a square. Consider a variable point P inside the square for which $\angle BAP \geq 60°$. Let Q be the intersection of the line AD and the perpendicular to BP in P. Let R be the intersection of the line BQ and the perpendicular to BP from C.

(a) Prove that $|BP| \geq |BR|$.

(b) For which point(s) P does the inequality in (a) become an equality?

Solution



We claim that ΔABP and ΔRCB are similar triangles. Indeed, if we denote the intersection of BP and CR by S, then $\angle RCB = \angle SCB = 90° - \angle SBC = 90° - \angle PBC = \angle ABP$. Moreover, the right angles in P and A imply that A and P lie on the circle with diameter $[BQ]$, so either $ABPQ$ or $AQBP$ is a cyclic, convex quadrilateral. In either case, A and Q lie on the same side of BP, so $\angle PAB = \angle PQB$. Since CR and PQ are perpendicular to BP, these lines are parallel and hence $\angle PQB = \angle CRB$. Together with $\angle PAB = \angle PQB$ and $\angle RCB = \angle ABP$, this implies the claim that $\Delta ABP \sim \Delta RCB$.

This similarity yields the equality $|AP|/|BR| = |BP|/|BC|$. Since $\angle BAP \geq 60°$, we get

$$\begin{aligned}
0 \leq (|AB| - |AP|)^2 &= |AB|^2 + |AP|^2 - 2 \cdot |AB| \cdot |AP| \\
&= |BP|^2 + 2(\cos \angle BAP - 1) \cdot |AB| \cdot |AP| \\
&\leq |BP|^2 - |AB| \cdot |AP| = |BP|^2 - |BR| \cdot |BP|.
\end{aligned}$$

This implies that $|BP| \geq |BR|$, as desired.

In order for equality to occur, one needs equality in each of the inequalities considered above: $|AB| = |AP|$ and $\angle BAP = 60°$. Hence there is exactly one point P for which we have equality; this is the unique point inside the square such that ΔABP is equilateral.