BxMO
2011

THIRD BENELUX
MATHEMATICAL OLYMPIAD

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PROBLEMS

AND

SOLUTIONS

Problem 1

An ordered pair of integers (m, n) with $1 < m < n$ is said to be a *Benelux couple* if the following two conditions hold: m has the same prime divisors as n, and $m + 1$ has the same prime divisors as $n + 1$.

(a) Find three Benelux couples (m, n) with $m \leqslant 14$.

(b) Prove that there exist infinitely many Benelux couples.

Solution

(a) It is possible to see that $(2, 8)$, $(6, 48)$ and $(14, 224)$ are Benelux couples.

(b) Let $k \geqslant 2$ be an integer and $m = 2^k - 2$. Define $n = m(m + 2) = 2^k(2^k - 2)$. Since m is even, m and n have the same prime factors. Also, $n + 1 = m(m + 2) + 1 = (m + 1)^2$, so $m + 1$ and $n + 1$ have the same prime factors, too. We have thus obtained a Benelux couple $\big(2^k - 2, 2^k(2^k - 2)\big)$ for each $k \geqslant 2$.

Problem 2

Let ABC be a triangle with incentre I. The angle bisectors AI, BI and CI meet $[BC]$, $[CA]$ and $[AB]$ at D, E and F, respectively. The perpendicular bisector of $[AD]$ intersects the lines BI and CI at M and N, respectively. Show that A, I, M and N lie on a circle.

Solution

The quadrilateral $AMDB$ is cyclic. Indeed, M is the intersection of the line BI, which bisects the angle $\widehat{ABD}$ in ABD and the perpendicular bisector of $[AD]$. By uniqueness of this intersection point, it follows that M lies on the circumcircle of ABD, and thence $AMDB$ is cyclic. Analogously, $ANDC$ is cyclic.

Moreover, $MNBC$ is cyclic, for $\widehat{NMI} = 90° - \widehat{EIA}$. Indeed, A and I lie on either side of the midpoint of $[AD]$, for $\widehat{BDA} = \widehat{CAD} + \widehat{BCA} > \widehat{BAD}$. But

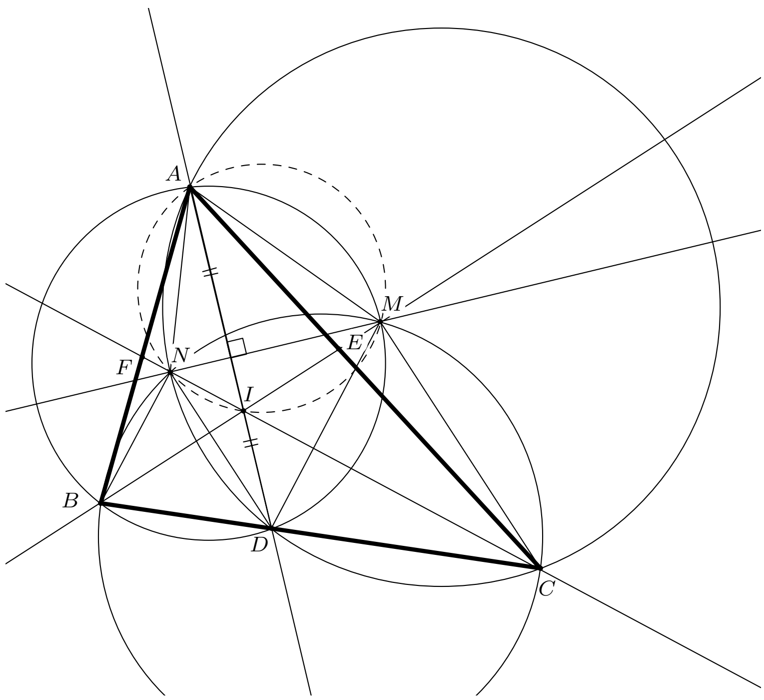
$$\widehat{EIA} = 180° - \tfrac{1}{2}\widehat{BAC} - \widehat{BEA} = 180° - \tfrac{1}{2}\widehat{BAC} - \left(\tfrac{1}{2}\widehat{CBA} + \widehat{BCA}\right) = 90° - \tfrac{1}{2}\widehat{BCA}$$

It follows that $\widehat{NMI} = \widehat{BCI}$, which implies that $MNBC$ is cyclic, as the points I and D lie on the same side of MN.

There are now two ways of completing the proof:

Solution 1 (using $AMDB$ and $MNBC$)
Since $AMDB$ is cyclic, $\widehat{MAI} = \widehat{MAD} = \widehat{MBD}$, as, by construction, B and M lie on either side of AD. Moreover, $\widehat{MBD} = \widehat{MBC} = \widehat{MNC}$ for $MNBC$ is cyclic. Thus $\widehat{MAI} = \widehat{MNI}$, so $AMIN$ is cyclic, for M and N lie on either side of AD.

Solution 2 (using $AMDB$ and $ANDC$)
Since $AMDB$ and $ANDC$ are cylic, $\widehat{AMI} + \widehat{ANI} = \widehat{AMB} + \widehat{ANC} = \widehat{ADB} + \widehat{ADC} = 180°$, because B and M, and C and N lie on either side of AD. Hence $AMIN$ is cyclic, for M and N lie on either side of AD.



Remark. It is moreover true that $BM \perp DN$ and $CN \perp DM$. Indeed, symmetry implies that the image J of I under reflection in MN lies on the circumcircle of DMN. Moreover, DI is a height of DMN, so the fact that I and J are equidistant from the side $[MN]$ implies that I is the orthocentre of DMN. This implies the claim.

Problem 3

If k is an integer, let $\mathrm{c}(k)$ denote the largest cube that is less than or equal to k. Find all positive integers p for which the following sequence is bounded:

$$a_0 = p \qquad \text{and} \qquad a_{n+1} = 3a_n - 2\mathrm{c}(a_n) \quad \text{for } n \geqslant 0.$$

Solution

Since $\mathrm{c}(a_n) \leqslant a_n$ for all $n \in \mathbb{N}$, $a_{n+1} \geqslant a_n$ with equality if and only if $\mathrm{c}(a_n) = a_n$. Hence the sequence is bounded if and only if it is eventually constant, which is if and only if a_n is a perfect cube, for some $n \geqslant 0$. In particular, the sequence is bounded if p is a perfect cube.

We now claim that, if a_n is not a cube for some n, then neither is a_{n+1}. Indeed, if a_n is not a cube, $q^3 < a_n < (q+1)^3$ for some $q \in \mathbb{N}$, so that $\mathrm{c}(a_n) = q^3$. Suppose to the contrary that a_{n+1} is a cube. Then

$$a_{n+1} = 3a_n - 2\mathrm{c}(a_n) < 3(q+1)^3 - 2q^3 = q^3 + 9q^2 + 9q + 3 < q^3 + 9q^2 + 27q + 27 = (q+3)^3$$

Also, since $\mathrm{c}(a_n) < a_n$, $a_{n+1} > a_n > q^3$, so $q^3 < a_{n+1} < (q+3)^3$. It follows that the only possible values of a_{n+1} are $(q+1)^3$ and $(q+2)^3$. However, in both of these cases,

$$3a_n - 2q^3 = a_{n+1} = (q+1)^3 \iff 3a_n = 3\big(q^3 + q^2 + q\big) + 1$$
$$3a_n - 2q^3 = a_{n+1} = (q+2)^3 \iff 3a_n = 3\big(q^3 + 2q^2 + 4q\big) + 8$$

a contradiction modulo 3. This proves that, if a_n is not a cube, then neither is a_{n+1}. Hence, if p is not a perfect cube, a_n is not a cube for any $n \in \mathbb{N}$, and the sequence is not bounded. We conclude that the sequence is bounded if and only if p is a perfect cube.

Problem 4

Abby and Brian play the following game: They first choose a positive integer N. Then they write numbers on a blackboard in turn. Abby starts by writing a 1. Thereafter, when one of them has written the number n, the other writes down either $n+1$ or $2n$, provided that the number is not greater than N. The player who writes N on the blackboard wins.

(a) Determine which player has a winning strategy if $N = 2011$.

(b) Find the number of positive integers $N \leqslant 2011$ for which Brian has a winning strategy.

Solution

(a) Abby has a winning strategy for odd N: Observe that, whenever any player writes down an odd number, the other player has to write down an even number. By adding 1 to that number, the first player can write down another odd number. Since Abby starts the game by writing down an odd number, she can force Brian to write down even numbers only. Since N is odd, Abby will win the game. In particular, Abby has a winning strategy if $N = 2011$.

(b) – Let $N = 4k$. If any player is forced to write down a number $m \in \{k+1, k+2, \ldots, 2k\}$, the other player wins the game by writing down $2m \in \{2k+2, 2k+4, \ldots, 4k\}$, for the players will have to write down the remaining numbers one after the other. Since there is an even number of numbers remaining, the latter player wins. This implies that the player who can write down k, i.e. has a winning strategy for $N = k$, wins the game for $N = 4k$.

– Similarly, let $N = 4k+2$. If any player is forced to write down a number $m \in \{k+1, k+2, \ldots, 2k+1\}$, the other player wins the game by writing down $2m \in \{2k+2, 2k+4, \ldots, 4k+2\}$, as in the previous case. Analogously, this implies that the player who has a winning strategy for $N = k$ wins the game for $N = 4k+2$.

Since Abby wins the game for $N = 1, 3$, while Brian wins the game for $N = 2$, Brian wins the game for $N = 8, 10$, as well, and thus for $N = 32, 34, 40, 42$, too. Then Brian wins the game for a further 8 values of N between 128 and 170, and thence for a further 16 values between 512 and 682, and for no other values with $N \leqslant 2011$. Hence Brian has a winning strategy for precisely 31 values of N with $N \leqslant 2011$.