BxMO 2011

THIRD BENELUX MATHEMATICAL OLYMPIAD

Luxembourg, 6–8 May 2011

Language: **English**

Problem 1. An ordered pair of integers (m, n) with $1 < m < n$ is said to be a *Benelux couple* if the following two conditions hold: m has the same prime divisors as n, and $m+1$ has the same prime divisors as $n+1$.

(a) Find three Benelux couples (m, n) with $m \leqslant 14$.

(b) Prove that there exist infinitely many Benelux couples.

Problem 2. Let ABC be a triangle with incentre I. The angle bisectors AI, BI and CI meet $[BC]$, $[CA]$ and $[AB]$ at D, E and F, respectively. The perpendicular bisector of $[AD]$ intersects the lines BI and CI at M and N, respectively. Show that A, I, M and N lie on a circle.

Problem 3. If k is an integer, let $\mathrm{c}(k)$ denote the largest cube that is less than or equal to k. Find all positive integers p for which the following sequence is bounded:

$$a_0 = p \qquad \text{and} \qquad a_{n+1} = 3a_n - 2\mathrm{c}(a_n) \quad \text{for } n \geqslant 0.$$

(*A sequence $a_0, a_1, \ldots$ of reals is said to be* bounded *if there exists $M \in \mathbb{R}$ such that, for all $n \geqslant 0$, $|a_n| \leqslant M$.*)

Problem 4. Abby and Brian play the following game: They first choose a positive integer N. Then they write numbers on a blackboard in turn. Abby starts by writing a 1. Thereafter, when one of them has written the number n, the other writes down either $n+1$ or $2n$, provided that the number is not greater than N. The player who writes N on the blackboard wins.

(a) Determine which player has a winning strategy if $N = 2011$.

(b) Find the number of positive integers $N \leqslant 2011$ for which Brian has a winning strategy.

Time: 4 hours and 30 minutes.
Each problem is worth 7 marks.