Solutions of Benelux Mathematical Olympiad 2010

Problem 1. A finite set of integers is called *bad* if its elements add up to 2010. A finite set of integers is a *Benelux-set* if none of its subsets is bad. Determine the smallest integer n such that the set $\{502, 503, 504, \ldots, 2009\}$ can be partitioned into n Benelux-sets.
(A partition of a set S into n subsets is a collection of n pairwise disjoint subsets of S, the union of which equals S.)

Solution. As $502 + 1508 = 2010$, the set $S = \{502, 503, \ldots, 2009\}$ is not a Benelux-set, so $n = 1$ does not work. We will prove that $n = 2$ does work, i.e. that S can be partitioned into 2 Benelux-sets.
Define the following subsets of S:

$$\begin{aligned}
A &= \{502, 503, \ldots, 670\},\\
B &= \{671, 672, \ldots, 1005\},\\
C &= \{1006, 1007, \ldots, 1339\},\\
D &= \{1340, 1341, \ldots, 1508\},\\
E &= \{1509, 1510, \ldots, 2009\}.
\end{aligned}$$

We will show that $A \cup C \cup E$ and $B \cup D$ are both Benelux-sets.
Note that there does not exist a bad subset of S of one element, since that element would have to be 2010. Also, there does not exist a bad subset of S of more than three elements, since the sum of four or more elements would be at least $502+503+504+505 = 2014 > 2010$. So any possible bad subset of S contains two or three elements.
Consider a bad subset of two elements a and b. As $a, b \geq 502$ and $a + b = 2010$, we have $a, b \leq 2010 - 502 = 1508$. Furthermore, exactly one of a and b is smaller than 1005 and one is larger than 1005. So one of them, say a, is an element of $A \cup B$, and the other is an element of $C \cup D$. Suppose $a \in A$, then $b \geq 2010 - 670 = 1340$, so $b \in D$. On the other hand, suppose $a \in B$, then $b \leq 2010 - 671 = 1339$, so $b \in C$. Hence $\{a, b\}$ cannot be a subset of $A \cup C \cup E$, nor of $B \cup D$.
Now consider a bad subset of three elements a, b and c. As $a, b, c \geq 502$, $a + b + c = 2010$, and the three elements are pairwise distinct, we have $a, b, c \leq 2010 - 502 - 503 = 1005$. So $a, b, c \in A \cup B$. At least one of the elements, say a, is smaller than $\frac{2010}{3} = 670$, and at least one of the elements, say b, is larger than 670. So $a \in A$ and $b \in B$. We conclude that $\{a, b, c\}$ cannot be a subset of $A \cup C \cup E$, nor of $B \cup D$.
This proves that $A \cup C \cup E$ and $B \cup D$ are Benelux-sets, and therefore the smallest n for which S can be partitioned into n Benelux-sets is $n = 2$. $\square$

Remark. Observe that $A \cup C \cup E_1$ and $B \cup D \cup E_2$ are also Benelux-sets, where $\{E_1, E_2\}$ is any partition of E.

Problem 2. Find all polynomials $p(x)$ with real coefficients such that

$$p(a+b-2c)+p(b+c-2a)+p(c+a-2b)=3p(a-b)+3p(b-c)+3p(c-a)$$

for all $a,b,c \in \mathbb{R}$.

Solution 1. For $a=b=c$, we have $3p(0)=9p(0)$, hence $p(0)=0$. Now set $b=c=0$, then we have

$$p(a)+p(-2a)+p(a)=3p(a)+3p(-a)$$

for all $a \in \mathbb{R}$. So we find a polynomial equation

$$p(-2x)=p(x)+3p(-x). \tag{1}$$

Note that the zero polynomial is a solution to this equation. Now suppose that p is not the zero polynomial, and let $n \geq 0$ be the degree of p. Let $a_n \neq 0$ be the coefficient of x^n in $p(x)$. At the left-hand side of (1), the coefficient of x^n is $(-2)^n \cdot a_n$, while at the right-hand side the coefficient of x^n is $a_n + 3 \cdot (-1)^n \cdot a_n$. Hence $(-2)^n = 1 + 3 \cdot (-1)^n$. For n even, we find $2^n = 4$, so $n=2$, and for n odd, we find $-2^n=-2$, so $n=1$. As we already know that $p(0)=0$, we must have $p(x)=a_2x^2+a_1x$, where a_1 and a_2 are real numbers (possibly zero).
The polynomial $p(x)=x$ is a solution to our problem, as

$$(a+b-2c)+(b+c-2a)+(c+a-2b)=0=3(a-b)+3(b-c)+3(c-a)$$

for all $a,b,c \in \mathbb{R}$. Also, $p(x)=x^2$ is a solution, since

$$\begin{aligned}(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2 &= 6(a^2+b^2+c^2)-6(ab+bc+ca)\\ &= 3(a-b)^2+3(b-c)^2+3(c-a)^2\end{aligned}$$

for all $a,b,c \in \mathbb{R}$.
Now note that if $p(x)$ is a solution to our problem, then so is $\lambda p(x)$ for all $\lambda \in \mathbb{R}$. Also, if $p(x)$ and $q(x)$ are both solutions, then so is $p(x)+q(x)$. We conclude that for all real numbers a_2 and a_1 the polynomial $a_2x^2+a_1x$ is a solution. Since we have already shown that there can be no other solutions, these are the only solutions. $\square$

Solution 2. For $a=b=c$, we have $3p(0)=9p(0)$, hence $p(0)=0$. Now set $b=c=0$, then we have

$$p(a)+p(-2a)+p(a)=3p(a)+3p(-a)$$

for all $a \in \mathbb{R}$. So we find a polynomial equation

$$p(-2x)=p(x)+3p(-x). \tag{2}$$

Define $q(x) = p(x) + p(-x)$, then we find that

$$q(2x) = p(2x) + p(-2x) = (p(-x) + 3p(x)) + (p(x) + 3p(-x)) = 4q(x). \tag{3}$$

Note that the zero polynomial is a solution to this equation. Now suppose that q is not the zero polynomial, and let $m \geq 0$ be the degree of q. Let $b_m \neq 0$ be the coefficient of x^m in $q(x)$. At the left-hand side of (3), the coefficient of x^m is $2^m \cdot b_m$, while at the right-hand side the coefficient of x^m is $4b_m$. Hence $m = 2$. As $q(x) = p(x) + p(-x)$, the polynomial $q(x)$ does not contain any nonzero terms with odd exponent of x. Since also $q(0) = 2p(0) = 0$, we conclude that

$$q(x) = b_2 x^2,$$

where b_2 is a real number (possibly zero).
From (2) we now deduce that $p(2x) = p(-x) + 3p(x) = 2p(x) + q(x)$, so

$$p(2x) - 2p(x) = b_2 x^2. \tag{4}$$

Suppose that that degree n of p is greater than 2. Let $a_n \neq 0$ be the coefficient of x^n in $p(x)$. At the left-hand side of (4), the coefficient of x^n is $(2^n - 2) \cdot a_n \neq 0$. But the coefficient of x^n at the right-hand side vanishes, yielding a contradiction. So the degree of p is at most 2. As we already know that $p(0) = 0$, we must have $p(x) = a_2 x^2 + a_1 x$, where a_1 and a_2 are real numbers (possibly zero).
We finally check that every polynomial of this form is indeed a solution (see solution 1). $\square$

Problem 3. On a line l there are three different points A, B and P in that order. Let a be the line through A perpendicular to l, and let b be the line through B perpendicular to l. A line through P, not coinciding with l, intersects a in Q and b in R. The line through A perpendicular to BQ intersects BQ in L and BR in T. The line through B perpendicular to AR intersects AR in K and AQ in S.

(a) Prove that P, T, S are collinear.

(b) Prove that P, K, L are collinear.

Solution 1.

(a) Since P, R and Q are collinear, we have $\triangle PAQ \sim \triangle PBR$, hence

$$\frac{|AQ|}{|BR|} = \frac{|AP|}{|BP|}.$$

Conversely, P, T and S are collinear if it holds that

$$\frac{|AS|}{|BT|} = \frac{|AP|}{|BP|}.$$

So it suffices to prove

$$\frac{|BT|}{|BR|} = \frac{|AS|}{|AQ|}.$$

Since $\angle ABT = 90° = \angle ALB$ and $\angle TAB = \angle BAL$, we have $\triangle ABT \sim \triangle ALB$. And since $\angle ALB = 90° = \angle QAB$ and $\angle LBA = \angle ABQ$, we have $\triangle ALB \sim \triangle QAB$. Hence $\triangle ABT \sim \triangle QAB$, so

$$\frac{|BT|}{|BA|} = \frac{|AB|}{|AQ|}.$$

Similarly, we have $\triangle ABR \sim \triangle AKB \sim \triangle SAB$, so

$$\frac{|BR|}{|BA|} = \frac{|AB|}{|AS|}.$$

Combining both results, we get

$$\frac{|BT|}{|BR|} = \frac{|BT|/|BA|}{|BR|/|BA|} = \frac{|AB|/|AQ|}{|AB|/|AS|} = \frac{|AS|}{|AQ|},$$

which had to be proved.

(b) Let the line PK intersect BR in B_1 and AQ in A_1 and let the line PL intersect BR in B_2 and AQ in A_2. Consider the points A_1, A and S on the line AQ, and the points B_1, B and T on the line BR. As $AQ \parallel BR$ and the three lines A_1B_1, AB and ST are concurrent (in P), we have

$$A_1A : AS = B_1B : BT,$$

where all lengths are directed. Similarly, as A_1B_1, AR and SB are concurrent (in K), we have

$$A_1A : AS = B_1R : RB.$$

This gives

$$\frac{BB_1}{BT} = \frac{RB_1}{RB} = \frac{RB + BB_1}{RB} = 1 + \frac{BB_1}{RB} = 1 - \frac{BB_1}{BR},$$

so

$$BB_1 = \frac{1}{\frac{1}{BT} + \frac{1}{BR}}.$$

Similary, using the lines A_2B_2, AB and QR (concurrent in P) and the lines A_2B_2, AT and QB (concurrent in L), we find

$$B_2B : BR = A_2A : AQ = B_2T : TB.$$

This gives

$$\frac{BB_2}{BR} = \frac{TB_2}{TB} = \frac{TB + BB_2}{TB} = 1 + \frac{BB_2}{TB} = 1 - \frac{BB_2}{BT},$$

so

$$BB_2 = \frac{1}{\frac{1}{BR} + \frac{1}{BT}}.$$

We conclude that $B_1 = B_2$, which implies that P, K and L are collinear.

$\square$

Solution 2.

(a) Define X as the intersection of AT and BS, and Y as the intersection of AR and BQ. To prove that P, S and T are collinear, we will use Menelaos' theorem in $\triangle ABX$, so we have to prove

$$\frac{AP}{PB}\frac{BS}{SX}\frac{XT}{TA} = -1.$$

Note that B is between P and A, X is between S and B, and X is between T and A, so it suffices to prove that

$$\frac{|AP|}{|PB|}\frac{|BS|}{|SX|}\frac{|XT|}{|TA|} = 1.$$

Because AQ and BR are parallel, we have $\triangle AQP \sim \triangle BRP$, hence

$$\frac{|AP|}{|BP|} = \frac{|QA|}{|RB|}. \tag{5}$$

Also, since $\angle ASB = \angle KBR$ and $\angle BAS = 90^\circ = \angle BKR$, we have $\triangle ASB \sim \triangle KBR$, hence

$$\frac{|BS|}{|RB|} = \frac{|AS|}{|KB|}, \quad \text{so} \quad |BS| = \frac{|AS|}{|KB|}|RB|. \tag{6}$$

Similarly, we have $\triangle ATB \sim \triangle QAL$, hence

$$\frac{|TA|}{|AQ|} = \frac{|TB|}{|AL|}, \quad \text{so} \quad |TA| = \frac{|TB|}{|AL|}|AQ|. \tag{7}$$

As $\angle ASX = \angle ASB = 90^\circ - \angle ABS = 90^\circ - \angle ABK = \angle KAB = \angle YAB$, and $\angle SAX = 90^\circ - \angle XAB = 90^\circ - \angle LAB = \angle ABL = \angle ABY$, we have $\triangle SXA \sim \triangle AYB$, hence

$$\frac{|SX|}{|AY|} = \frac{|AS|}{|BA|}, \quad \text{so} \quad |SX| = \frac{|AS|}{|BA|}|AY|. \tag{8}$$

Similarly, we have $\triangle BXT \sim \triangle AYB$, hence

$$\frac{|XT|}{|YB|} = \frac{|BT|}{|AB|}, \quad \text{so} \quad |XT| = \frac{|BT|}{|AB|}|YB|. \tag{9}$$

By combining (5) - (9), we find

$$\begin{aligned} \frac{|AP|}{|PB|}\frac{|BS|}{|SX|}\frac{|XT|}{|TA|} &= \frac{|QA|}{|RB|} \cdot \frac{|AS|}{|KB|}|RB| \cdot \frac{|BA|}{|AS||AY|} \cdot \frac{|BT|}{|AB|}|YB| \cdot \frac{|AL|}{|TB||AQ|} \\ &= \frac{|AL|}{|KB|}\frac{|YB|}{|AY|}. \end{aligned} \tag{10}$$

Since $\angle YLA = 90^\circ = \angle YKB$ and $\angle AYL = \angle BYK$, we have $\triangle AYL \sim \triangle BYK$, hence

$$\frac{|AL|}{|BK|} = \frac{|AY|}{|BY|}, \quad \text{so} \quad \frac{|AL|}{|BK|}\frac{|BY|}{|AY|} = 1. \tag{11}$$

By combining (10) and (11), we find

$$\frac{|AP|}{|PB|}\frac{|BS|}{|SX|}\frac{|XT|}{|TA|} = 1,$$

as we wanted to prove.

(b) Again, we will use Menelaos' theorem in $\triangle ABX$, so we have to prove

$$\frac{AP}{PB}\frac{BK}{KX}\frac{XL}{LA} = -1.$$

Note that $\frac{AP}{PB} < 0$, and $\frac{BK}{KX} < 0$ if and only if $\frac{XL}{LA} < 0$, so it suffices to prove that

$$\frac{|AP|}{|PB|}\frac{|BK|}{|KX|}\frac{|XL|}{|LA|} = 1.$$

As $\angle BXL = \angle AXK$ and $\angle BLX = 90° = \angle AKX$, we have $\triangle BLX \sim \triangle AKX$, hence

$$\frac{|XL|}{|XK|} = \frac{|BL|}{|AK|}. \tag{12}$$

Since $\angle ALB = 90° = \angle QAB$, we have $\triangle ALB \sim \triangle QAB$, hence

$$\frac{|LA|}{|AQ|} = \frac{|LB|}{|AB|}, \quad \text{so} \quad |LA| = \frac{|LB|}{|AB|}|AQ|. \tag{13}$$

Similarly, we have $\triangle AKB \sim \triangle ABR$, hence

$$\frac{|BK|}{|RB|} = \frac{|AK|}{|AB|}, \quad \text{so} \quad |BK| = \frac{|AK|}{|AB|}|RB|. \tag{14}$$

By combining (5) and (12) - (14), we find

$$\frac{|AP|}{|PB|}\frac{|BK|}{|KX|}\frac{|XL|}{|LA|} = \frac{|QA|}{|RB|} \cdot \frac{|BL|}{|AK|} \cdot \frac{|AB|}{|LB||AQ|} \cdot \frac{|AK|}{|AB|}|RB| = 1,$$

which is what we wanted to prove.

□

Solution 3. As $\angle AKB = \angle ALB = 90°$, the points K and L belong to the circle with diameter AB. Since $\angle QAB = \angle ABR = 90°$, the lines AQ and BR are tangents to this circle.

Apply Pascal's theorem to the points A, A, K, L, B and B, all on the same circle. This yields that the intersection Q of the tangent in A and the line BL, the intersection R of the tangent in B and the line AK, and the intersection of KL and AB are collinear. So KL passes through the intersection of AB and QR, which is point P. Hence P, K and L are collinear. This proves part b.

Now apply Pascal's theorem to the points A, A, L, K, B and B. This yields that the intersection S of the tangent in A and the line BK, the intersection T of the tangent in B and the line AL, and the intersection P of KL and AB are collinear. This proves part a. □

Solution 4.

(a) W.l.o.g. we may assume that $A=(0,0)$ and $B=(1,0)$ and the line through P is in the upper half plane, so l is the x-axis, a is the y-axis and b is the line $x=1$. Take $P=(p,0)$ $(p>1)$ and $Q=(0,q)$ $(q>0)$. Since PQ is given by $\frac{x}{p}+\frac{y}{q}=1$, we find $R=(1,\frac{q(p-1)}{p})$.

Now AR is given by $y=\frac{q(p-1)}{p}x$, hence BS, the line perpendicular to AR and passing through $B=(1,0)$, is given by $y=-\frac{p}{q(p-1)}(x-1)$. We find $S=(0,\frac{p}{q(p-1)})$.

Moreover BQ is given by $y=-q(x-1)$, hence AT, the line perpendicular to BQ and passing through $A=(0,0)$, is given by $y=\frac{1}{q}x$. We find $T=(1,\frac{1}{q})$. Since $\frac{|BT|}{|BP|}=\frac{1/q}{p-1}=\frac{\frac{p}{q(p-1)}}{p}=\frac{|AS|}{|AP|}$, we conclude that P, T and S are collinear.

(b) Point K is the intersection of AR and BS. Solving for x yields

$$\begin{aligned}\frac{q(p-1)}{p}x &= -\frac{p}{q(p-1)}(x-1)\\ \left(\frac{q(p-1)}{p}+\frac{p}{q(p-1)}\right)x &= \frac{p}{q(p-1)}\\ x &= \frac{\frac{p}{q(p-1)}}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}\end{aligned}$$

so

$$K=\left(\frac{\frac{p}{q(p-1)}}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}\ ,\ \frac{1}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}\right).$$

Point L is the point of intersection of AT and BQ. Solving for x yields

$$\begin{aligned}\frac{1}{q}x &= -q(x-1)\\ \left(\frac{1}{q}+q\right)x &= q\\ x &= \frac{q}{\frac{1}{q}+q}\end{aligned}$$

so

$$L=\left(\frac{q}{\frac{1}{q}+q}\ ,\ \frac{1}{\frac{1}{q}+q}\right).$$

Let K_0 and L_0 be the projections of K and L on the x-axis. We have to show that the following fractions are equal:

$$\frac{|K_0K|}{|K_0P|}=\frac{\frac{1}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}}{p-\frac{\frac{p}{q(p-1)}}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}}\qquad\text{and}\qquad\frac{|L_0L|}{|L_0P|}=\frac{\frac{1}{\frac{1}{q}+q}}{p-\frac{q}{\frac{1}{q}+q}}$$

Working out cross products twice, this comes down to

$$
\begin{aligned}
\frac{1}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}} \cdot \left(p-\frac{q}{\frac{1}{q}+q}\right) &\stackrel{?}{=} \frac{1}{\frac{1}{q}+q} \cdot \left(p-\frac{\frac{p}{q(p-1)}}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}\right) \\
\left(\frac{1}{q}+q\right) \cdot \left(p-\frac{q}{\frac{1}{q}+q}\right) &\stackrel{?}{=} \left(\frac{q(p-1)}{p}+\frac{p}{q(p-1)}\right) \cdot \left(p-\frac{\frac{p}{q(p-1)}}{\frac{q(p-1)}{p}+\frac{p}{q(p-1)}}\right) \\
\frac{p}{q}+pq-q &\stackrel{?}{=} q(p-1)+\frac{p^2}{q(p-1)}-\frac{p}{q(p-1)} \\
\frac{p}{q}+pq-q &\stackrel{?}{=} q(p-1)+\frac{p(p-1)}{q(p-1)},
\end{aligned}
$$

which is clearly true. $\square$

Problem 4. Find all quadruples (a, b, p, n) of positive integers, such that p is a prime and

$$a^3 + b^3 = p^n.$$

Solution 1. Let (a, b, p, n) be a solution. Note that we can write the given equation as

$$(a + b)(a^2 - ab + b^2) = p^n.$$

As a and b are positive integers, we have $a+b \geq 2$, so $p \mid a+b$. Furthermore, $a^2-ab+b^2 = (a-b)^2 + ab$, so either $a = b = 1$ or $a^2 - ab + b^2 \geq 2$. Assume that the latter is the case. Then p is a divisor of both $a+b$ and a^2-ab+b^2, hence also of $(a+b)^2-(a^2-ab+b^2) = 3ab$. This means that p either is equal to 3 or is a divisor of ab. Since p is a divisor of $a+b$, we have $p \mid a \Leftrightarrow p \mid b$, hence either $p = 3$, or $p \mid a$ and $p \mid b$. If $p \mid a$ and $p \mid b$, then we can write $a = pa'$, $b = pb'$ with a' and b' positive integers, and we have $(a')^3 + (b')^3 = p^{n-3}$, so $(a', b', p, n-3)$ then is another solution (note that $(a')^3 + (b')^3$ is a positive integer greater than 1, so $n - 3$ is positive).
Now assume that (a_0, b_0, p_0, n_0) is a solution such that $p \nmid a$. From the reasoning above it follows that either $a_0 = b_0 = 1$, or $p_0 = 3$. After all, if we do not have $a_0 = b_0 = 1$ and we have $p_0 \neq 3$, then $p \mid a$. Also, given an arbitrary solution (a, b, p, n), we can divide everything by p repeatedly until there are no factors p left in a.
Suppose $a_0 = b_0 = 1$. Then the solution is $(1, 1, 2, 1)$.
Suppose $p_0 = 3$. Assume that $3^2 \mid (a_0^2 - a_0b_0 + b_0^2)$. As $3^2 \mid (a_0 + b_0)^2$, we then have $3^2 \mid (a_0 + b_0)^2 - (a_0^2 - a_0b_0 + b_0^2) = 3a_0b_0$, so $3 \mid a_0b_0$. But $3 \nmid a_0$ by assumption, and $3 \mid a_0 + b_0$, so $3 \nmid b_0$, which contradicts $3 \mid a_0b_0$. We conclude that $3^2 \nmid (a_0^2 - a_0b_0 + b_0^2)$. As both $a_0 + b_0$ and $a_0^2 - a_0b_0 + b_0^2$ must be powers of 3, we have $a_0^2 - a_0b_0 + b_0^2 = 3$. Hence $(a_0 - b_0)^2 + a_0b_0 = 3$. We must have $(a_0 - b_0)^2 = 0$ or $(a_0 - b_0)^2 = 1$. The former does not give a solution; the latter gives $a_0 = 2$ and $b_0 = 1$ or $a_0 = 1$ and $b_0 = 2$.
So all solutions with $p \nmid a$ are $(1, 1, 2, 1)$, $(2, 1, 3, 2)$ and $(1, 2, 3, 2)$. From the above it follows that all other solutions are of the form $(p_0^k a_0,\ p_0^k b_0,\ p_0,\ n_0 + 3k)$, where (a_0, b_0, p_0, n_0) is one of these three solutions. Hence we find three families of solutions:

- $(2^k,\ 2^k,\ 2,\ 3k + 1)$ with $k \in \mathbb{Z}_{\geq 0}$,
- $(2 \cdot 3^k,\ 3^k,\ 3,\ 3k + 2)$ with $k \in \mathbb{Z}_{\geq 0}$,
- $(3^k,\ 2 \cdot 3^k,\ 3,\ 3k + 2)$ with $k \in \mathbb{Z}_{\geq 0}$.

It is easy to check that all these quadruples are indeed solutions. $\square$

Solution 2. Let (a, b, p, n) be a solution. Note that we can write the given equation as

$$(a + b)(a^2 - ab + b^2) = p^n.$$

As a and b are positive integers, we have $a + b \geq 2$ and $a^2 - ab + b^2 = (a - b)^2 + ab \geq 1$. So both factors are positive and therefore must be powers of p. Let k be an integer with $1 \leq k \leq n$ such that $a + b = p^k$. Then $a^2 - ab + b^2 = p^{n-k}$. If we substitute $b = p^k - a$, we find

$$p^{n-k} = (a + b)^2 - 3ab = p^{2k} - 3a(p^k - a).$$

We can rewrite this as:

$$3a^2 - 3p^k a + p^{2k} - p^{n-k} = 0,$$

from which we see that a is a solution of the following quadratic equation in x:

$$3x^2 - 3p^k x + p^{2k} - p^{n-k} = 0. \tag{15}$$

The discriminant of (15) is

$$D = (-3p^k)^2 - 4 \cdot 3 \cdot (p^{2k} - p^{n-k}) = 3 \cdot (4p^{n-k} - p^{2k}) = 3p^{n-k} \cdot (4 - p^{3k-n}).$$

As $p^{n-k} = (a + b)^2 - 3ab < (a + b)^2 = p^{2k}$, we have $n - k < 2k$, so $3k - n > 0$. Since a is a solution of (15), the discriminant must be nonnegative. Hence $4 - p^{3k-n} \geq 0$. If $p = 2$, this implies $3k - n = 1$ or $3k - n = 2$; if $p = 3$, this implies $3k - n = 1$; and if $p > 3$, then $p \geq 5$ so $4 \geq p^{3k-n}$ can never be true.
Suppose $p = 2$ and $3k - n = 1$. Then $D = 3 \cdot 2^{2k-1} \cdot (4 - 2) = 3 \cdot 2^{2k}$. But this is a not a square, so the solutions of (15) will not be integers, which yields a contradiction.
Suppose $p = 2$ and $3k - n = 2$. Then $D = 3 \cdot 2^{2k-2} \cdot (4 - 4) = 0$, so the only solution of (15) is $x = \frac{3 \cdot 2^k}{2 \cdot 3} = 2^{k-1}$. Therefore $a = 2^{k-1}$ and $b = 2^k - a = 2^{k-1}$, and this gives a solution for all $k \geq 1$, namely $(2^{k-1}, 2^{k-1}, 2, 3k - 2)$.
Suppose $p = 3$ and $3k - n = 1$. Then $D = 3 \cdot 3^{2k-1} \cdot (4 - 3) = 3^{2k}$, so the solutions of (15) are $x = \frac{3^{k+1} \pm 3^k}{2 \cdot 3} = \frac{1}{2}(3^k \pm 3^{k-1})$. Therefore $a = 2 \cdot 3^{k-1}$ or $a = 3^{k-1}$. For all $k \geq 1$ we find the solutions $(2 \cdot 3^{k-1}, 3^{k-1}, 3, 3k - 1)$ and $(3^{k-1}, 2 \cdot 3^{k-1}, 3, 3k - 1)$.
We conclude that there are three families of solutions:

- $(2^{k-1}, 2^{k-1}, 2, 3k - 2)$ with $k \in \mathbb{Z}_{\geq 1}$,
- $(2 \cdot 3^{k-1}, 3^{k-1}, 3, 3k - 1)$ with $k \in \mathbb{Z}_{\geq 1}$,
- $(3^{k-1}, 2 \cdot 3^{k-1}, 3, 3k - 1)$ with $k \in \mathbb{Z}_{\geq 1}$.

It is easy to check that all these quadruples are indeed solutions. $\square$